Time-dependent, irreversible Entropy Production and Geodynamics

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We present an application of entropy production as an abstraction tool for complex processes in geodynamics. We relate the thermodynamic theory to the underlying energy conservation principles and their application to continuum mechanics. In plasticity theory extremum principles of plastic work (upper and lower bound principles) have long proven to be extremely useful in understanding basic modes of plastic deformation of materials. The principles acknowledge the fact that an infinite number of velocity fields can satisfy a given plastic stress field. The principles allow a meaningful workaround avoiding the necessity to come up with complete solutions, which are often difficult to obtain. We show that these principles have been derived without the consideration of the explicit role of time. The fundamental processes of geodynamic deformation are inherently time-dependent. While we face the same problems as in the plastic deformation of metals, we have to consider the explicit role of non-linear entropy production through time. We propose a way forward that allows a basic assessment of modes of geodynamic deformation on the basis of extrema in entropy production. These principles revert to the classical limit theorems in the time independent case.

Keywords: Limit Theorems, Plasticity Theory, Finite Time Thermodynamics, irreversible Entropy, Carnot Efficiency, endoreversible engine

Introduction

We hypothesize that minimum and maximum entropy production can be used as an abstraction tool to understand basic modes of geodynamic processes. The advantage of this approach is that we can predict basic planetary behaviour without going into specific modelling of the underlying physics. We suggest that extremum principles in entropy production are an extremely useful first pass before embarking on laborious modelling of complex geodynamic processes.

Earlier work on this subject comes from the chemical modelling or the climate modelling community focusing either on the minimum e.g. [Glansdorff et al., 1973] or the maximum principle of entropy production e.g. [Paltridge, 1979] . Although these approaches allow very elegant solutions to complicated problems in Earth Sciences [Ozawa et al., 2003] they are nonetheless entirely based on min/max assumptions about how a thermodynamic system should behave. No reasons are given why the thermodynamic system should behave one way or the other. We show that a natural fluid dynamic heat transfer system switches from a minimum of entropy production defined by Fourier's law conduction to an increased production of entropy upon the onset of convection. We introduce a non-dimensional number to characterize steady state entropy production rate. The arguments are derived from the classic paper of Tolman and Fine [1948] but here we leave out their explicit incorporation of chemistry.

Energy Balance

Classical continuum mechanics is based on physical laws that are cast in such a way that energy flows are compatible with the first and second laws of thermodynamics. The fundamental notions of conservation of energy and increase of entropy are hence firmly embedded during a mechanical process. We consider a representative volume element (RVE) subjected to external loads. The application of the principle of virtual power results in:

$$\wp_{\text{int}} + \wp_{ext} = \wp_{acc} \tag{1}$$

where the internal power, $\wp_{\text{int}}$, is the product of the internal forces, - $\underline{\underline{B}}_{ij}$, with the virtual velocities, $\underline{\underline{v}}_{ij}$, integrated over the RVE. This internal power is balanced by the virtual power of the exterior and inertial forces $\wp_{ext}$ and $\wp_{acc}$, respectively. Applying the first law of thermodynamics gives:

$$\frac{d}{dt}\int_V \rho u\, dV = \tilde{W} + \int_V r_i\, dV - \int_{\partial V} \underline{q}\cdot\underline{n}\, d\Gamma \tag{2}$$

Where u is the specific internal energy $\underline{q}$ is a heat flux vector, $\underline{n}$ is a normal vector to the surface, Γ, of the RVE, $\tilde{W} = \int \underline{\underline{B}}_{ij} : \underline{\underline{v}}_{ij} dV$ is the rate of total dissipative work. We use the "tilde" for the work rate to emphasize that time integrals are path dependent. This is the classic linear irreversible thermodynamic formulation in the case of thermo-mechanical, quasi-static equilibrium. More details can be found in [Regenauer-Lieb et al., 2009]. The source term, r_i is a discrete summation which collects the rate of heat production (e.g. radioactive decay, Joule heating, chemical reaction) inside the RVE. This additional source term has been criticized [Houlsby and Puzrin, 2007]. The term r_i is only necessary as an "artificial" source term in the energy equation if the effect on the internal energy is not resolved by the thermodynamic calculation. The source term would not be necessary if the change in isotopes through radioactive decay or chemical changes or electrical currents inside the RVE are explicitly resolved or calculated in a full equation of state.

Therefore, incorporating equations (1) and (2), Gauss' Theorem, the conservation of mass, and the assumption of an arbitrary *V*, at the local level we obtain the energy rate equation for the RVE in thermal-mechanical, quasi-static equilibrium as

$$\rho\dot{u} = \underline{\underline{B}}_{ij} : \underline{\underline{v}}_{ij} + r_i - div\underline{q} \tag{3}$$

where the over dot identifies the material time derivative and : signifies the double dot (scalar contraction) product of two tensors. This description of the internal energy balances out the mechanical and thermal energies coming from the mechanically induced heat source contribution from

$\underline{\underline{B_{ij}}} : \underline{\underline{v_{ij}}}$ in quasi-static equilibrium, with the additional solution-strategy dependent term r_i and heat fluxes.

Limit Theorems in Continuum Mechanics

In continuum mechanics, solutions to quasi-static deformation are sought where time dependent thermodynamic fluxes have relaxed to steady state. This condition is normally ensured by making use of the maximum rate of dissipation or maximum entropy production principle [*Ziegler*, 1983]. For such time-independent solutions upper and lower bounds of the energy required to deform a solid can be obtained from limit theorems described next.

Limit theorems in continuum mechanics have originally been put forward for solving three groups of differential equation of a non-hardening plastic-rigid material. These are the stress equilibrium equations, the stress-strain relations and the equations relating strain and displacement. There is no unique solution for the plastic material under consideration. An infinity of statically admissible stress states will satisfy the stress boundary conditions, the equilibrium equations and the yield criterion alone. In addition, an infinite number of virtual kinematically admissible displacement modes will be compatible with a continuous distortion satisfying the displacement boundary conditions [Bishop, 1953]. Therefore, one has to be satisfied with partial or incomplete solutions for which limit theorems have been put forward to provide assessment of the predicted yield point loads [Hill, 1951].

Lower bound Theorem: The yield-point loads determined from a distribution of stress that satisfies the equilibrium equations and stress boundary conditions and which nowhere violates the yield criterion, are not greater than the actual yield-point loads. In other words, these yield loads produce the minimum of plastic work on the system because the plastic displacements are zero.

Upper bound Theorem: The yield-point loads, determined by equating the external work to the internal work done in a deformation mode satisfying the displacement boundary conditions by that stress distribution needed to enforce those conditions at each point, are not less than the actual yield-point loads. The stress distribution need not be in equilibrium and is only defined where the strain-rate is non-zero. In other words the upper bound only considers the dissipative part of the deformation process, which according to Ziegler's principles uses maximum entropy production as a thermodynamic consistency check. The limit theorems do not, however, involve in themselves an explicit thermodynamic treatment.

These theorems have since proven invaluable for engineering calculations of a wide field of rheologies and have been extended over a wide range of applications [Salencon, 2002]. They provide sound estimates for the upper and lower bounds of energy required to deform solids. We propose here that an extension to the entropy balance will provide similar use in geodynamics.

Entropy Balance

Geodynamic problems are inherently irreversible and time dependent. We define the entropy production, in accordance with Tolman and Fine [1948, equation 18.4], by:

$$\tilde{S} = \int_V \frac{r_i}{T} dV - \int_{\partial V} \frac{q}{T} \cdot \underline{n} d\Gamma + \tilde{S}_{irr} \qquad (4)$$

Hence the internal irreversible entropy generation collects the thermo-chemical dissipative sources. With this definition the entropy, a local form can be obtained in an analogy with the derivation of the energy equation (3):

$$\rho T \, \tilde{s} = \rho T \, \tilde{s}_{irr} + r_i - T div\left(\frac{q}{T}\right) \qquad (5)$$

where $\tilde{s}$ is the path dependent total specific entropy change rate in the local form, $\tilde{s}_{irr}$ is the irreversible specific entropy production rate, and T the absolute temperature. We recognize the first term on the right hand side as the entropy production caused by mechanical, chemical, radioactive, and electrical process contributions, which must be positive or zero according to the second law of thermodynamics. By the additional constraint that the absolute temperature $T > 0$ the entropy balance law encapsulates the entire system of classical laws of thermodynamics [Colli et al., 2005]. The third term is the total entropy flux, which homogenizes the entropy change rate. This concept of writing the second law of thermodynamics in an equality form and the point that the entropy rather than the temperature describes the underlying physical processes are fundamental points of Tolman and Fine's [1948] classic paper.

Minimum and Maximum Entropy Production

Consider a thermodynamic system that from an initial time t_{init} reaches a given final state in time t_{fin}. We assume this final state may be relaxed to an equilibrium state, or it may be a steady state, which can only be sustained by an internal dissipation process.

We identify two extrema that relate to whether the entire system reaches equilibrium (minimum entropy production) or whether it is in a steady state, which could be sustained by an internal dissipation process (entropy production is not a minimum). The important role of entropy production has been proposed by Tolman and Fine [1948] in their equation 5.1. We reformulate their arguments that allow changes in internal temperature through time. For this we invoke the Helmholtz formulation of the internal energy.

$$u = \psi + sT \tag{6}$$

where ψ is the specific Helmholtz free energy dependent on the state variables. We differentiate equation (6) with respect to time and obtain

$$\dot{u} = \dot{\psi} + \tilde{s}T + \dot{T}s \tag{7}$$

According to equations (1) and integrating equation (7) we can formulate an efficiency equation for the thermodynamic system. This efficiency is defined as the total amount of recoverable power $\dot{J} = \int \frac{\partial \dot{\psi}}{\partial \underline{\underline{\alpha_{ij}}}} : \partial \underline{\underline{\dot{\alpha}_{ij}}} dV$ that can be stored in the thermodynamic system, where $\partial \underline{\underline{\alpha_{ij}}}$ describes all recoverable state variables, like elastic strain. This stored power is limited by two bounds

$$\dot{J}_{\max} = \max\left[\tilde{W} - \left\{\int_V \frac{q \cdot grad(T)}{T} dV + \int_V \rho T \,\tilde{s}_{irr}\, dV\right\}\right] \quad (8)$$

$$\dot{J}_{\min} = \min\left[\tilde{W} - \left\{\int_V \frac{q \cdot grad(T)}{T} dV + \int_V \rho T \,\tilde{s}_{irr}\, dV\right\}\right] \quad (9)$$

If we attribute the irreversible entropy to the plastic dissipation, assume Ziegler's principle of maximum dissipation and ignore the temperature feedback terms encapsulated in the time derivative of the irreversible entropy production [Regenauer-Lieb et al. 2009], equation (9) reverts to the classical upper bound principle. The classical lower bound considers only the work and neglects effects of temperature and time derivatives. Our formulation extends this by adding the second and third term in equation (8). Therefore we hope to be able to extend the *yield design theory* underpinned by the limit theorems in continuum mechanics [Salencon, 2002] for discrete examples in the future. Also, if we assume the temperature of the environment T_{surf} to be constant we can use an explicit open system formulation and from integrating equation (8) over finite time we recover the basic equation for *finite time thermodynamics* [*Andresen*, 1995].

Note that the analogy of the limit theorems in terms of work can only be cast rigorously into a limit theorem of dissipated power and not integrated for work. The integration of equation (8) or (9) is only possible over finite time for specific classes of thermodynamic problems such as cyclical or steady state problems as shall be discussed below. In general non-equilibrium situations work or heat are not properties of the system.

Limit Theorems in Geodynamics

The above considerations of minimum/maximum work or maximum/minimum entropy production, respectively apply to engineering applications where we may apply some kind of engineering design (e.g. optimal control theory) to achieve a desired goal. The approach lends itself to immediately extend the limit theorems of continuum mechanics to a time dependent process.

We are left with the discussion of how meaningful an extrapolation of the entropy production extrema and the related upper and lower bounds is for geodynamics. We do not answer the critical question whether nature really seeks an extremum in quasi-steady state but instead we discuss the merits of the extremum principles using the insights from thermodynamics to provide upper and lower bounds to general deformation processes that must not be violated. We also show the sensitivity of the solution of entropy production at the large scale boundaries of our system and inside the system. From these examples it will become clear that we need to consider the explicit role of entropy production for time dependent deformation processes such as models of mantle convection.

We now discuss a simple cyclical dissipative system for which the efficiency equations (8) and (9) can be integrated over finite time. Note that this integration of the weak form is not universally valid for any thermodynamic process since work and heat are imperfect differentials. We can integrate by considering a temperature T_0 as the temperature of the environment (which in our case is chosen to be the surface temperature of the Earth) as an additional constraint. We reformulate equations (8) and (9) in the strong form and obtain the following formulation for the recoverable (elastic) energy rate,

$$\dot{J} = \tilde{W} - \int_V \left(\frac{T - T_0}{T} \right) div\underline{q}\, dV - \int_V \left(\frac{T - T_0}{T_0} \right) T_0 \tilde{s}\, dV - T_0 \int_V \frac{\underline{q} \cdot \underline{\underline{grad}}(T)}{T^2} dV - \int_V \rho T_0 \tilde{s}_{irr}\, dV$$

(10)

where $\tilde{W}$ is the total work rate, the second term on the right hand side identifies the equivalent work rate generated due the net heat input at T , the third term is the equivalent work rate consumed to upgrade the entropy change rate of the system from T_0 to T and the last two terms are the irreversibilties due heat transfer and chemo-mechanical processes, respectively. Thermodynamicists will recognize the second term as the work of a Carnot engine and the third term as the work of a Carnot refrigerator. The strong formulation for the recoverable power in equation (10) is valid for any thermodynamic system relaxing to cyclic or steady state equilibrium although, the following example considers only a restrictive cyclical process. In particular equation (10) is valid for chaotic heat transfer process as long as a continuum formulation is valid.

For finite time integration we assume that during each cycle there is no net change of the recoverable work through change in state variables $\partial \underline{\underline{\alpha_{ij}}}$ like the elastic strain or similar and $\dot{J} = 0$. Also the Carnot refrigerator work term is zero under cyclic steady state. The system is only driven by the amount of heat absorbed per cycle *Q*. The fourth term on the right hand side of equation (10) is nonzero for heat transfer in a spherical Earth, however, for a linear temperature profile problem it is zero. Equation (10) can then be integrated to yield the following expression.

$$W = \frac{T_{core} - T_{surf}}{T_{core}} Q - T_0 \Delta S_{irr} \qquad (11)$$

By way of this Carnot analogy [Pons and Le Quere, 2005], the maximum efficiency of the Earth's heat engine is the theoretical maximum Carnot limit $\eta = \frac{T_{core} - T_{surf}}{T_{core}} = 1 - \frac{T_{surf}}{T_{core}}$ where the temperature of the environment is $T_0 = T_{surf}$ being the surface temperature and T_{core} the temperature at the core mantle boundary. The irreversible entropy production cannot be above this limit and therefore it defines a theoretical upper bound of the amount of work that we can extract from the efficiency equation (10). We develop a generalized interpretation of equation (10) in terms of minimum and maximum entropy production between two surfaces

Upper and Lower Bounds of dissipation to self driven heat transfer phenomena

The thermodynamic upper and lower bounds to self driven heat transfer phenomena between two surfaces at dissimilar temperatures can be indexed by a non-dimensional entropy generation number introduced here. For obtaining bounds to a quasi-incompressible medium we revisit equation (10) and define a measure of rate of the irreversibilities $\tilde{H}$, which is also known as lost work for the case of the linear temperature gradient. A minimum in the lost work corresponds to a minimum in irreversibility of the system and surroundings.

$$\tilde{H} \equiv \int_V \left(\frac{T - T_0}{T_0} \right) T_0 \tilde{s}\, dV + \int_V \rho T_0 \tilde{s}_{irr}\, dV + \dot{J} - \tilde{W} \qquad (12)$$

it follows from equation (10) that

$$\tilde{H} = -\int_V \left(\frac{T - T_0}{T} \right) div\underline{q}\, dV - T_0 \int_V \frac{\underline{q} \cdot grad(T)}{T^2} dV \qquad (13)$$

Consider an Earth where we have only observations available at the surface and do not know whether conduction or convection occurs. We assume boundary layers to be outside of our system. We have to consider two possible cases of heat transfer, which are considered in parallel, conduction and convection (Figure 1). The case where all heat transfer is given by conduction and no convection occurs is the minimum entropy production case.

For definition of a nondimensional entropy generation number ζ that uses the Carnot engine as the base case but also applies to irreversible engines we normalize the lost work with the equivalent surface conductive heat transfer

$$\zeta \equiv \frac{\tilde{H}}{Q_{cond}} \qquad (14)$$

where k is the thermal conductivity and l is the distance between the surface and the core. The maximum entropy production for the Carnot case with any internal temperature profile can hence be described by the Carnot efficiency

$$\zeta = Nu \left(1 - \frac{T_{surf}}{T_{core}} \right) = Nu\,\eta \qquad (15)$$

where Nu is the Nusselt number defined by convective over conductive heat transfer. The smallest entropy generation for the maximum entropy production defined by the Carnot case is the limit where the Nusselt number $Nu \rightarrow 1$. For $Nu=1$ the contribution of maximum entropy production defined by the Carnot limit is suspended and the minimum entropy production is defined by conduction.

Under the assumption of classical finite-time thermodynamics the Nusselt number cannot be directly derived because it is controlled by transport properties such as viscosity. The incorporation of these transport properties, requires consideration of multi-scale couplings in the rate of entropy generation, which appear as additional time and space dependent entropy feedback terms [Dahmen et al., 2009; Regenauer-Lieb et al., 2009] We can obtain theoretical bounds for the Nusselt number for incompressible fluids by postulating that the second time derivative of this entropy generation is zero.

We do not consider this case here. Our analysis includes, however, the limits for work related heat production (shear heating) because we assume that all the heat from the core is transferred to the surface and the total entropy of the system does not change regardless of where the entropy is produced. The upper limit on entropy production can, however, be changed if we consider the inclusion of radiogenic heat production at arbitrary locations inside the mantle.

Introduction of Radiogenic Heat

Radiogenic heat production at arbitrary locations inside the mantle is here considered with following constraints:

$$\begin{aligned} \frac{Q_{surf}}{T_{surf}} &= \frac{Q_{core}}{T_{core}} + \frac{Q_{rad}}{\hat{T}} \\ \frac{1}{T_{surf}} &= \frac{\Phi}{T_{core}} + \frac{1-\Phi}{\hat{T}} \end{aligned} \qquad (16)$$

where $\hat{T}^{-1} = \left(\hat{1/T} \right)_{rad} = \int_V \frac{r_i}{T} dV \Big/ \int_V r_i \, dV$ is the dimensional temperature stemming from the normalization of integration of the radiogenic heat term r_i over the mantle to deliver the radiogenic heat term Q_{rad} and Φ the nondimensionalization of the heat by division through Q_{surf}. The second constraint is that

$$W = Q_{surf} + Q_{core} + Q_{rad} \tag{17}$$

With the definition of the three reservoir efficiency measure

$$\eta_{3C} = \frac{W}{Q_{core} + Q_{rad}} \tag{18}$$

we obtain

$$\eta_{3C} = 1 - \frac{T_{surf}}{T_{core}}\Phi - \frac{T_{surf}}{\hat{T}}\left(1 - \Phi\right) = \eta\Phi - \frac{T_{surf}}{\hat{T}}\left(1 - \Phi\right) \tag{19}$$

replacing η in equation (15) and leads to a total dissipation as discussed in Figure 2.

Application of limit analysis to a Cartesian Earth Model

By way of example assume that at the largest scale of geodynamic simulations the Earth is a reversible heat engine where mantle convection performs a generalized Carnot cycle. We assume here that all the mechanical work done that drives plate tectonic is converted into heat in the interior of the Earth. We also assume that the Earth is a Cartesian box with a perfectly flat surface (no potential energy change at the surface). This concept describes a certain class of computer models of mantle convection where heat is supplied and extracted at the boundaries and internal irreversible losses through radioactive decay are neglected. Similarly, all irreversible losses attributed to coupling of the Earth to its environment are neglected.

Both minimum entropy production cases are trivial: Case I is an isothermal Earth, when $T_{core} \rightarrow T_{surf}$ and case II is the Earth as purely conductive medium. Both cases are obviously not very attractive for a realistic Earth model.

The maximum upper bound of an irreversible engine – not necessarily a Carnot engine - is given by three engines with three heat reservoirs. The first two realizations are illustrated in Figures 2a and b are irreversible heat engines. The third realization in Figure 2c, involves a Carnot heat engine that is constructed to deposit its work into shear heating at either surface or core mantle boundary which includes an arbitrary internal radiogenic heat source. Figure 2c is an interesting case. While depositing the work as shear heating at the surface may not be very realistic, if instead the work were deposited at the lithosphere-asthenosphere boundary the idealized Carnot analogy may be a very good zero order approximation to mantle convection. In summary, all three maximum entropy production cases are more attractive solutions than the lower bound. The solutions show the necessity of explicitly calculating the conversion of mechanical work into heat in any mantle convection problem. We now turn to a more quantitative analysis.

Assume a core-mantle boundary temperature between 4500-3600 K [Hirose et al., 2007; Knittle and Jeanloz, 1991] and a constant surface temperature 300 K the value of the maximum Carnot efficiency is between 0.93-0.92. Note that this estimate is solely derived just from the assumption of surface and core-mantle temperature. Like the Carnot efficiency it provides a fundamental thermodynamic constraint that surprisingly, does not involve assumptions about material parameters such as rheology and thermal material parameters or geometric characterizations. This value is a fundamental nonimensional dissipation number related to an upper thermodynamic bound on the efficiency of heat transfer through a convecting mantle.

Discussion

We have shown that thermodynamics provides an abstraction tool for assessing the validity of complex numerical simulations of the Earth. Thermodynamic limits can be used as a simple means to assess the validity of numerical simulations and thermodynamic approaches also provide indications on possible oversights in numerical approaches. A good example is the neglect of link between mechanical deformation and heat, i.e. the dissipation caused by shear heating in the mantle. Through this neglect it is possible to calculate efficiencies higher than the Carnot limit which are absolutely incompatible with basic thermodynamics [Pons and Le Quere, 2005]. We have discussed here two bounds: maximum irreversible entropy production (convection with maximum entropy production) and the other extreme minimum entropy production (conduction). Both bounds are thermodynamically permissible.

Another important stimulus provided by the thermodynamic approaches is to look at the Earth boundary conditions for the mantle convection simulations in a critical way. As the next level of refinement a more realistic scenario is to consider the Earth not as an ideal Carnot engine heat engine but consider the replacement of the boundaries with irreversible losses by an endoreversible engine in future numerical models. These endoreversible boundaries could be identified as the conductive lithosphere and the convective outer core (Figure 3). For such an engine the Carnot limit is much reduced. The endoreversible limit for an Earth with irreversible heat transfer losses with the environment is reduced to a value of $\eta_{endo} = 1 - \sqrt{\frac{T_{surf}}{T_{ICB}}}$ [Curzon and Ahlborn, 1975]. Again taking a surface temperature of 300 K and an inner core boundary temperature of T_{ICB} = 5600K [Alfe et al., 2002] the efficiency of plate-tectonics in an endoreversible Earth is thus reduced to η_{endo} = 0.76. Note that this estimate is solely derived just from the assumption of surface and inner core-outer core temperature as the boundary temperatures for the endoreversible boundaries. It provides a fundamental thermodynamic constraint that again does not involve assumptions about material parameters such as rheology and thermal material parameters or geometric characterizations. This value should not be violated by convection codes for the Earth mantle that are based on the endoreversible idealization.

The next step of refinement for thermodynamic description of the Earth as a heat engine would be to include a parallel leaky wire (conduction) to the Curzon-Ahlborn engine as for the cases in Figures 1 - 3. This case has the interesting property that the maximum efficiency does not coincide with the maximum entropy production [Gordon and Huleihil, 1992].

We believe that finite time thermodynamics has a lot to offer to geodynamics through assessment of basic concepts such as thermodynamic length and constant thermodynamic speed. The concept of following explicitly the irreversible entropy production has been specifically identified as one of the key problems to be solved in the science of unstable Earth phenomena such as earthquakes [Ben Zion, 2008; Regenauer-Lieb and Yuen, 2008], ice quakes and landslides [Regenauer-Lieb et al., 2009].

We have used mantle convection as the first example for the application of the extremum principles proposed here. Other simple applications that lend themselves to be interpreted by the Carnot analogy would be the geodynamo in the outer core e.g. [Stacey and Davis, 2008]. In our basic example for the Earth as a Carnot heat engine we have found that the assumption of minimum entropy production identifies the thermodynamic equilibrium system with minimally dissipative heat conduction. This is a possible solution for low heat input with subcritical Rayleigh number. However, when convection starts and reaches steady state the assumption of maximum entropy production provides an excellent upper bound of the mode of heat transfer. Without calculation of the critical Rayleigh number we are in no position to decide between the two extremes. Note that when the Rayleigh number is raised to the chaotic regime the notion of maximum entropy production as a good estimate of the heat transfer mode for steady state breaks down [Castillo and Hoover, 1998]. Time-dependent convective instabilities are also possible under lower Rayleigh numbers if highly non-linear rheologies are used [Moresi and Solomatov, 1998]. The upper and lower bounds provide a fundamental measure for predicting thermodynamically permissible bounds of steady state convection calculations but should not be interpreted beyond steady state.

We conclude with the possible speculations on proposed limit theorems for other applications. The strongest result of this paper is the new extension of limit theorems in continuum mechanics, which are appended by the ones proposed here for entropy production. Future work will extend the concept for more realistic models of the Earth in terms of the analogy of an irreversible heat engine which will tighten the permissible bounds even further. We also will investigate the concept of time-dependent, irreversible entropy production for designing new thermodynamically based steady-state rheologies for use in Geodynamics and Continuum Mechanics.

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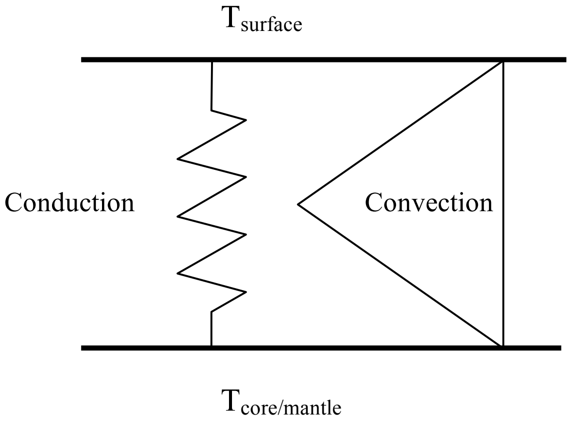
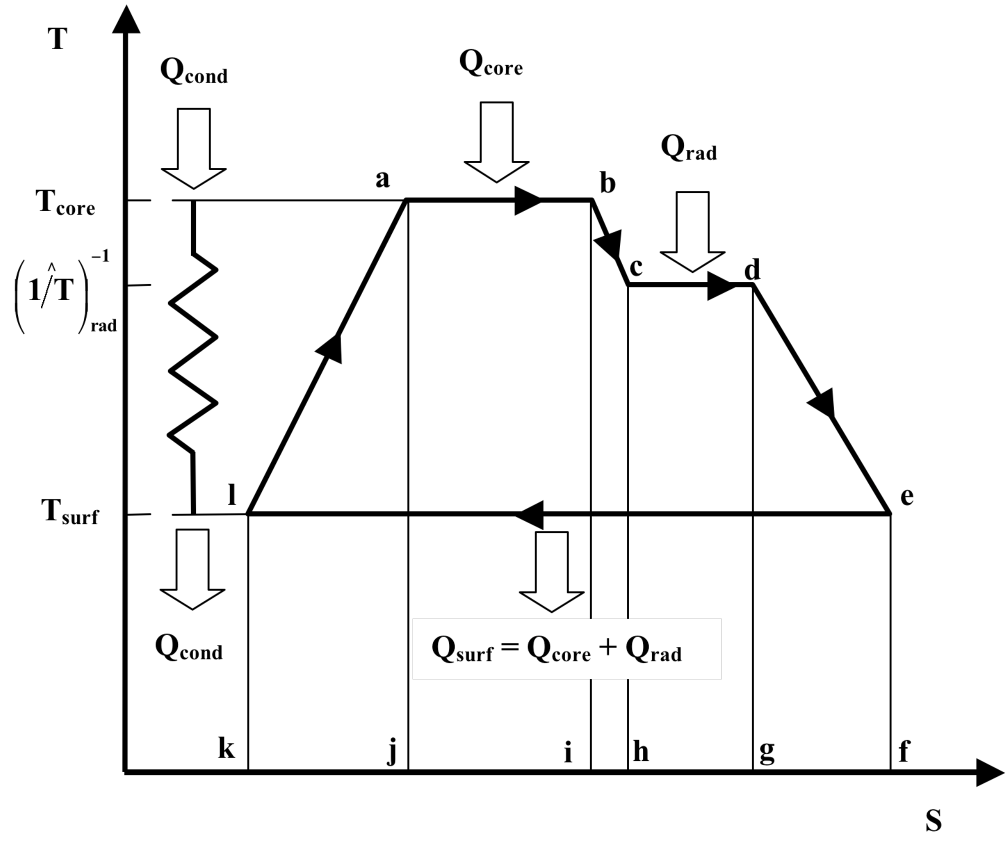


Figure 1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Q_{cond}
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l
Q_{cond}
$Q_{surf} = Q_{core} + Q_{rad}$
k
j
i
h
g
f
S

Figure 2a

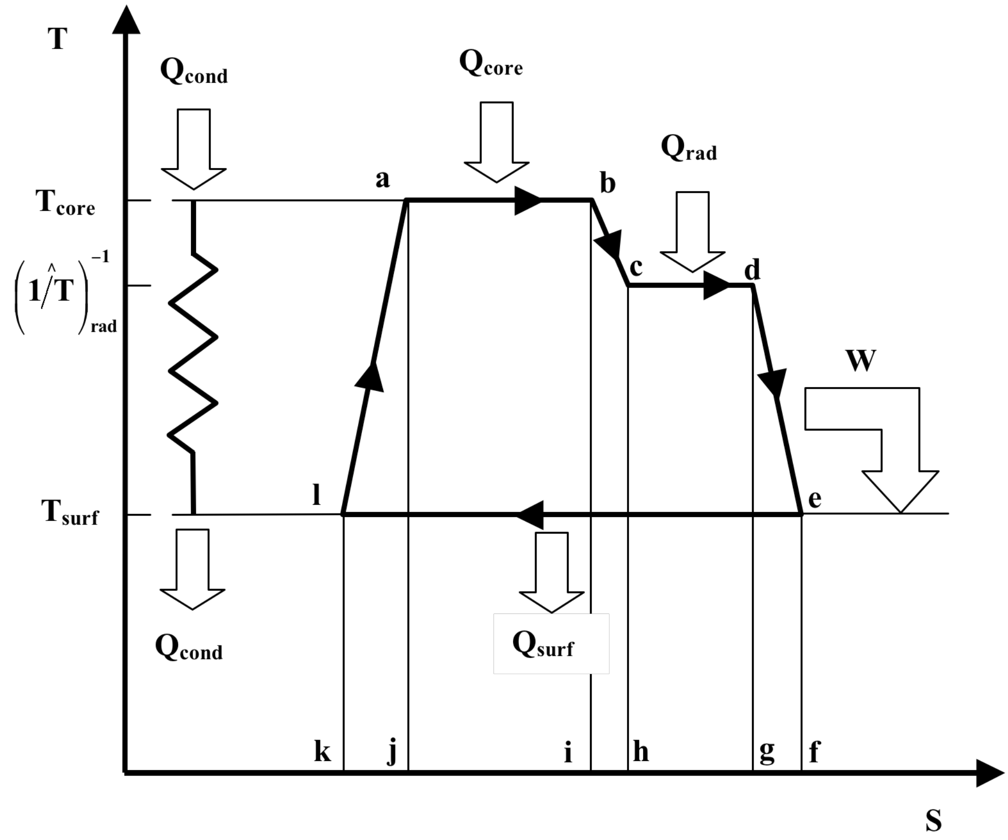
T
Q_cond
Q_core
Q_rad
T_core
$\left(\hat{1/T}\right)^{-1}_{rad}$
a
b
c
d
W
T_surf
l
e
Q_cond
Q_surf
k
j
i
h
g
f
S

Figure 2b

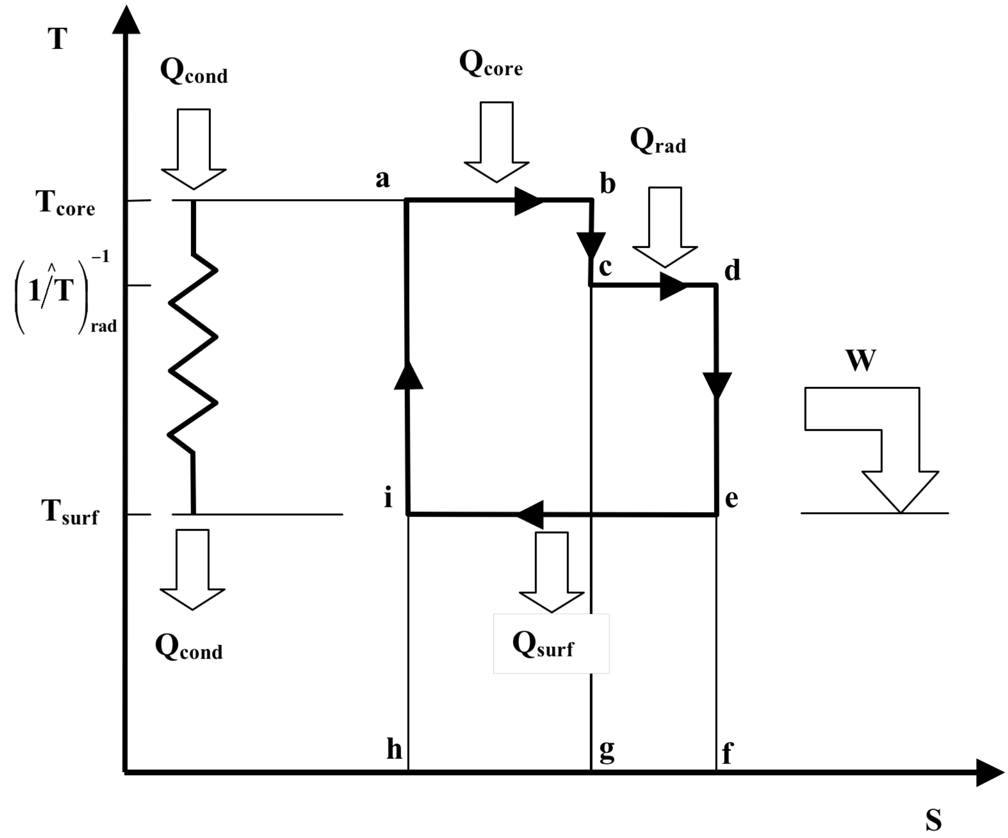
T
Q_cond
Q_core
Q_rad
T_core
(1/T̂)^-1 rad
a
b
c
d
W
T_surf
i
e
Q_cond
Q_surf
h
g
f
S

Figure 2c

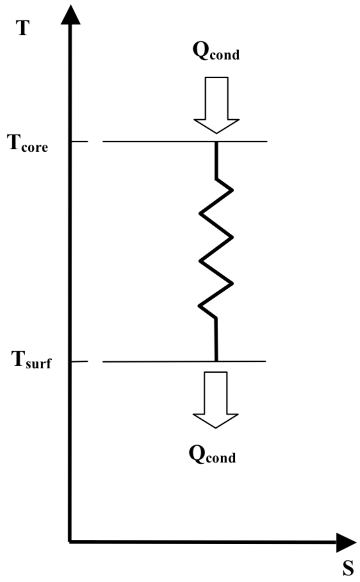

Figure 3

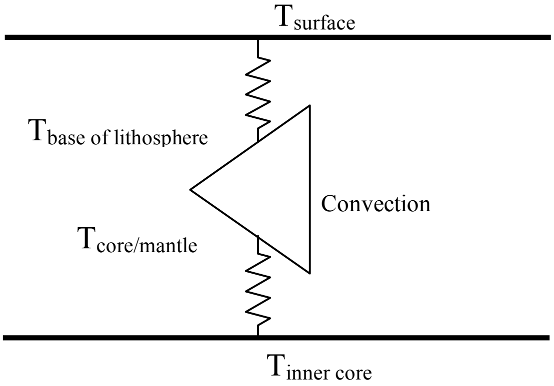

Figure 4

Figure captions

Figure 1. Generalized model for planetary convection vs. conduction. We assume that an observer on a planetary surface only has knowledge of the surface temperature and no insights from seismological data. The observer cannot discern between a conductive and a convective case. We assume a parallel setup with the possibility of multiple layers of conduction and convection.

Figure 2. Maximally dissipative modes of self driven heat transfer from the core-mantle boundary to the surface with an affiliated lost work given by $\eta_{2C} Q_{cond} + \eta_{3C} \left(Q_{core} + Q_{rad} \right)$, where $\eta_{2C} = 1 - T_{surf} / T_{core}$ is the two-reservoir Carnot efficiency and $\eta_{3C} = 1 - \phi T_{surf} / T_{core} - (1 - \phi) T_{surf} \left(\widehat{1/T} \right)_{rad}$ is the three-reservoir Carnot efficiency, where ϕ is defined as $Q_{core} / \left(Q_{core} + Q_{rad} \right)$ and $\left(\widehat{1/T} \right)_{rad}$ is the effective temperature from the integration of the radiogenic heat source . (a) A maximally dissipative heat engine which generates zero work with a concomitant conduction heat leak path. Q_{core} or the heat received from the core-mantle boundary and Q_{rad} or the heat received from radioactive decays are totally transmitted through viscous dissipation and internal heat transfer dissipation as Q_{surf} so that the total amount of heat appearing at the surface is $Q_{core}+Q_{rad}+Q_{cond}$. Note that the area of the rectangle a-b-i-j, i.e. Q_{core} plus the area of the rectangle c-d-g-h, i.e. Q_{rad} equal to the area of the rectangle e-f-k-l or Q_{surf}. Process paths b-c, d-e and l-a are adiabats. (b) A partially dissipative heat engine which generates work with a concomitant conduction heat leak path. Q_{surf} is now less than $Q_{core}+Q_{rad}$, and the resulting work or W is dissipated as heat at the surface through shearing action, so that the total amount of heat appearing at the surface is still equal to $Q_{core}+Q_{rad}+Q_{cond}$. Note that the area of the rectangle a-b-i-j, i.e. Q_{core}, plus the area of the rectangle c-d-g-h, i.e. Q_{rad}, is larger than the area of the rectangle e-f-k-l or Q_{surf}. Process paths b-c, d-e and l-a are adiabats. (c) A three-reservoir Carnot heat engine which generates maximum work with a concomitant conduction heat leak path. Q_{surf} is the minimum heat rejection by the engine as dictated by the Second Law and W is dissipated as heat at the surface through shearing action, so that the total amount of heat appearing at the surface is $Q_{core}+Q_{rad}+Q_{cond}$.

Figure 3. A minimally dissipative mode of heat transfer from the core-mantle boundary temperature to the surface temperature with an affiliated lost work of $\eta_{2C}Q_{cond}$, where $\eta_{2C} = 1 - T_{surf}/T_{core}$ is the two-reservoir Carnot efficiency.

Figure 4. A schematic diagram of the endoreversible engine identifying the irreversible losses at the upper boundary to the lithosphere and the irreversible losses at the lower boundary to the geodynamo.